JOURNAL OF COMPUTATIONAL PHYSICS 4, 250–257 (1969)

Solution of a Certain Class of Nonlinear Two-Point Boundary Value Problems[1]

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ABSTRACT

The methods of quasilinearization and invariant imbedding are combined to form an effective numerical algorithm for solution of the two-point boundary value problem $du/dz = f(u, v, z)$, $-dv/dz = g(u, v, z)$, $u(0) = a$, $v(x) = y$, $0 \leqslant z \leqslant x$. The method is applied to several examples and seems to be fast, accurate, and simple to use. It can be generalized and extended in various ways.

1. INTRODUCTION

The method of invariant imbedding has been investigated extensively over the last decade as a technique of both analytical and computational interest in the study of two-point boundary value problems. While its origins lie in transport theory, the device has been applied to many problems arising in other fields. However, most of the problems successfully attacked have been linear in structure. When the imbedding method is applied to nonlinear problems, the resulting equations are usually nonlinear partial differential equations [*1*]. Frequently these equations appear at least as difficult to handle as the original problem and the advantages that the method provides in treating many linear problems are no longer present.

To circumvent this difficulty, we combine the techniques of invariant imbedding and quasilinearization. The latter device reduces certain classes of nonlinear prob-

[1] Research supported in part by the U.S. Atomic Energy Commission, Project Themis, and National Science Foundation Grant GP-5967.

lems to sequences of linear problems. Invariant imbedding is applied to each linear member of the sequence. Under appropriate conditions the corresponding sequence of solutions approaches the solution of the original problem. The fundamental invariant imbedding equation at each stage is the familiar Riccati equations whose numerical solution is often very well-behaved. (This is a principal advantage of the imbedding method.) The entire scheme appears to be accurate, fast, and easy to program. It is easy to see that the method converges any time the quasilinearization technique employed by itself converges.

The idea of combining the two methods is not new. Such a combination has been used in chemical engineering studies [2]. It has also been tried, in somewhat different form, on some sample problems at RAND [3].

The method of invariant imbedding is in a state of continuing development and has many ramifications. Those aspects of the method used in this paper have many points of contact with the device referred to in the Russian literature by the name of simple factorization or the sweep method (see [14]). Also, the form of quasilinearization we have employed is similar to Newton's method (see [5]).

2. The Algorithm

We consider the two-point boundary value problem

$$\frac{du}{dz} = f(u, v, z),$$

$$-\frac{dv}{dz} = g(u, v, z), \tag{2.1}$$

$$u(0) = a, \qquad v(x) = y, \qquad 0 \leqslant z \leqslant x.$$

If (2.1) is quasilinearized, equation (2.1) is replaced by

$$\begin{aligned} \frac{du_{n+1}}{dz} &= A_n(z)\, u_{n+1} + B_n(z)\, v_{n+1} + S_n^{+}(z), \\ -\frac{dv_{n+1}}{dz} &= C_n(z)\, u_{n+1} + D_n(z)\, v_{n+1} + S_n^{-}(z), \end{aligned} \tag{2.2a}$$

$$u_{n+1}(0) = a, \qquad v_{n+1}(x) = y, \qquad 0 \leqslant z \leqslant x, \tag{2.2b}$$

where

$$\begin{aligned} A_n(z) &= f_u(u_n\,, v_n\,, z), \\ B_n(z) &= f_v(u_n\,, v_n\,, z), \end{aligned}$$

$$\begin{aligned} C_n(z) &= g_u(u_n, v_n, z), \\ D_n(z) &= g_v(u_n, v_n, z), \\ S_n^+(z) &= f(u_n, v_n, z) - u_n f_u(u_n, v_n, z) \\ &\quad - v_n f_v(u_n, v_n, z), \\ S_n^-(z) &= g(u_n, v_n, z) - u_n g_u(u_n, v_n, z) \\ &\quad - v_n g_v(u_n, v_n, z). \end{aligned} \tag{2.3}$$

As mentioned in §1, for a significant class of problems the functions u_n and v_n converge to the solutions u and v of (2.1), at least provided u_0 and v_0 are reasonably well chosen (see [6]).

We now use invariant imbedding on (2.2) and write

$$u_n(z) = R_n(z)\, v_n(z) + Q_n(z). \tag{2.4}$$

The functions R_{n+1} and Q_{n+1} are then known to satisfy [4, 7]

$$\begin{aligned} \frac{dR_{n+1}}{dz} &= B_n(z) + [A_n(z) + D_n(z)]\, R_{n+1}(z) + C_n(z)\, R_{n+1}^2(z), \\ R_{n+1}(0) &= 0, \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} \frac{dQ_{n+1}}{dz} &= [A_n(z) + C_n(z)\, R_{n+1}(z)]\, Q_{n+1}(z) + R_{n+1}(z)\, S_n^-(z) + S_n^+(z), \\ Q_{n+1}(0) &= a. \end{aligned} \tag{2.6}$$

Equations (2.5) and (2.6) are the basic imbedding equations.

Let us consider how the above equations may be used. We suppose all functions to be known at n. We then integrate the Riccati equation (2.5) from $z = 0$ to $z = x$. Next Q_{n+1} is found from (2.6). Using (2.4) in (2.2b) yields

$$\begin{aligned} -\frac{dv_{n+1}}{dz} &= C_n(z)[R_{n+1}(z)\, v_{n+1}(z) + Q_{n+1}(z)] + D_n(z)\, v_{n+1}(z) + S_n^-(z), \\ v_{n+1}(x) &= y. \end{aligned} \tag{2.7}$$

Equation (2.7) can now be integrated from $z = x$ to $z = 0$. Finally, u_{n+1} is obtained algebraically from (2.4). (In practice, equations (2.5) and (2.6) are most easily treated as a system.)

The choice of u_0 and v_0 is left open. Rather obvious is the possibility

$$\begin{aligned} u_0(z) &\equiv a, \\ v_0(z) &\equiv y. \end{aligned} \tag{2.8}$$

Partial knowledge of the solution of (2.1) or of some of its properties may be helpful in selecting u_0 and v_0. Indeed, such knowledge may be vital in obtaining convergence.

The algorithm we have developed involves only three integrations at each iteration and hence seems to have advantages over similar ones (see [2]). As pointed out earlier, the Riccati equation (2.5) is usually well behaved numerically.

3. Some Numerical Results

In this section we present three examples to illustrate the numerical technique described above. We begin with a brief discussion of the numerical procedures involved in implementing this method. An IBM 360 Model 44, using double precision arithmetic, was used for all computations. A fourth order Runge-Kutta scheme was used for the numerical integration of equations (2.5), (2.6) and (2.7) with a step size varying from .001 to .01 depending on the particular problem.

Past experience has shown that the imbedding method is most useful when applied to a family of problems of different "lengths" x. In our examples we therefore introduce the family

$$\frac{du}{dz} = f(u, v, z),$$

$$-\frac{dv}{dz} = g(u, v, z), \tag{3.1}$$

$$u(0) = a, \qquad v(x_i) = y, \qquad i = 1, 2,..., N,$$

$$0 < x_1 < x_2 < \cdots < x_N .$$

Denote the solution of the system (3.1) on $[0, x_i]$ by $u^i(z)$, $v^i(z)$. For $i = 1$ the sequences $\{u_n^1(z)\}$ and $\{v_n^1(z)\}$ were generated using the initial iterates $u_0^1(z)$ and $v_0^1(z)$ as given by (2.8). For $i = k$, $2 \leqslant k \leqslant N$, the sequences $\{u_n^k(z)\}$ and $\{v_n^k(z)\}$ were generated using

$$u_0^k(z) = \begin{cases} u^{k-1}(z), & 0 \leqslant z \leqslant x_{k-1} \\ u^{k-1}(x_{k-1}), & x_{k-1} \leqslant z \leqslant x_k , \end{cases} \tag{3.2a}$$

$$v_0^k(z) = \begin{cases} v^{k-1}(z), & 0 \leqslant z \leqslant x_{k-1} \\ y, & x_{k-1} \leqslant z \leqslant x_k . \end{cases} \tag{3.2b}$$

For $x_k - x_{k-1}$ small, $u_0^k(z)$ and $v_0^k(z)$, as defined in (3.2), were quite close to the solutions $u^k(z)$ and $v^k(z)$ respectively and the iterates converged quite rapidly.

We now proceed to discuss our examples.

Example I. Perhaps the most classical nonlinear equation on which to try our numerical method is

$$\frac{d^2\psi}{dz^2} = e^{\psi}.$$

We pose this in the form

$$\frac{du}{dz} = v,$$

$$-\frac{dv}{dz} = -e^{u}, \tag{3.3}$$

$$u(0) = 0, \qquad v(x) = y.$$

The analytic solution is given implicitly by

$$\sqrt{2}\, z - \frac{2}{c} \log \left\{ (c + \sqrt{1 + c^2}) \left(\frac{e^{u(z)/2}}{c + e^{u(z)} + c^2} \right) \right\} = 0, \tag{3.4a}$$

and

$$\sqrt{2}\, z - \frac{2}{|c|} \left\{ \cos^{-1} \left(\frac{|c|}{e^{u(z)/2}} \right) - \cos^{-1} |c| \right\} = 0. \tag{3.4b}$$

Here c is of course determined in each case by using the boundary condition $v(x) = y$ and the relationship

$$\tfrac{1}{2} v(x)^2 = e^{u(x)} + c^2,$$

which may be obtained by a simple manipulation using (3.3).

The numerical results obtained for $y = 0$ are given in Table I. (For convenience we give only the values of u and v at the boundary points, $z = 0$ and $z = x$.)

TABLE I

x	$u(x)$	$v(0)$
0.1	−.0050	−.0997
0.2	−.0197	−.1974
0.3	−.0434	−.2915
0.4	−.0751	−.3805
0.5	−.1137	−.4636
0.6	−.1578	−.5404
0.7	−.2063	−.6106
0.8	−.2581	−.6745
0.9	−.3123	−.7324
1.0	−.3681	−.7848

These results were checked against the analytical solution given by (3.4) and 4 digit agreement was obtained. The computations in the preceding table took about 3.4 minutes of computing time. This compares very favorably with other techniques available.

Example II. As a second example, we examine

$$\frac{du}{dz} = v - \epsilon uv,$$

$$-\frac{dv}{dz} = u - \epsilon uv, \qquad (3.5)$$

$$u(0) = 0, \qquad v(x) = y, \qquad \epsilon \geqslant 0.$$

Equation (3.5) describes a simple transport process involving binary fission and particle–particle interaction. The solution has been analyzed in [8]. If $\epsilon = 0$, the system becomes critical for $x = \pi/2$; if $\epsilon > 0$, there is no critical length regardless of the input y.

For $y = 5$, $\epsilon = .01$, we obtained the results in Table II, while Table III gives data for $y = 100$ and $\epsilon = .01$. The results are compatible with those in [8].

TABLE II

x	$u(x)$	$v(0)$
1.0	7.1197	8.6321
1.1	8.6395	9.9049
1.2	10.6016	11.6356
1.3	13.1988	14.0201
1.4	16.6703	17.3028
1.5	21.2172	21.6913

TABLE III

x	$u(x)$	$v(0)$
1.0	76.5862	137.7468
1.1	79.3445	135.1366
1.2	81.7244	132.6055
1.3	83.7870	130.2344
1.4	85.5818	127.9925
1.5	87.1498	125.8811

Example III. Finally, to test the method on an ill-behaved problem, we chose

$$\frac{du}{dz} = v,$$
$$-\frac{dv}{dz} = -u(1 + u^2), \tag{3.6}$$
$$u(0) = 0, \qquad v(x) = \sec^2 x, \qquad 0 \leqslant z < \pi/2.$$

The solution to (3.6) is

$$u(z) = \tan z,$$
$$v(z) = \sec^2 z.$$

Clearly for x near $\pi/2$, the problem presents a numerical challenge. Physically, we may think of $\pi/2$ as the "critical" length of a nonlinear transport problem, although there has been no effort to develop a realistic transport analogy. The possibility of using our device to investigate problems of criticality and cascading was influential in our overall study, however.

The data in Table IV were generated using an integration step size of .005. The scheme defined by equations (3.2) was used to begin each iteration ($k \geqslant 2$). The number of iterations which yielded a uniform pointwise error of less than 5×10^{-6} is recorded as a matter of interest.

TABLE IV

x	$u(x)$	$\tan x$	Number of Iterations
1.0	1.5574	1.5574	5
1.3	3.6022	3.6021	5
1.5	14.1073	14.1014	8
1.51	16.4654	16.4281	5
1.52	19.6856	19.6695	5
1.53	24.5293	24.4984	5
1.54	32.5326	32.4611	5
1.55	48.2975	48.0785	6

4. Summary and Conclusions

We have presented an algorithm for the numerical solution of a class of two point boundary problems which seems to be simple to apply, relatively fast and accurate.

Many generalizations and extensions come to mind at once. For example, the case in which $u(x)$ is assigned can be handled similarly [9]. Further, systems of differential equations can be treated by obvious modifications of the method.

Convergence criteria for the method are clearly closely linked to those for quasilinearization in general. As indicated in the examples, however, it is often possible in practice to improve convergence by studying a sequence of problems over larger and larger intervals. The algorithm is especially easy to use with such a procedure.

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